Analytical Comparison Between Time- and Frequency-Domain Techniques for Phase-Noise Analysis

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***Abstract*—In the literature, different techniques have been presented for the phase-noise analysis of free-running oscillator circuits. In order to give some insight into the relationships existing between them, an analytical comparison is carried out in this paper between three different approaches. Two of them are time-domain approaches, based on Floquet's theory and the impulse sensitivity function, respectively, and the third one is the carrier modulation approach, in frequency domain. The application of Floquet's theory enables the calculation of periodic sensitivity functions to the noise perturbations. Here, the possibility to determine these functions through harmonic balance is demonstrated. This allows applying the whole stochastic characterization of phase noise, obtained from time-domain analysis, to circuits simulated through harmonic balance. For illustration, calculations in a cubic-nonlinearity oscillator are presented.**

***Index Terms*—Harmonic-balance analysis, nonlinear oscillators, phase noise.**

I. Introduction

A FREE-RUNNING oscillator is described by a system of nonlinear differential equations, which do not explicitly depend on time. Due to this fact, the time origin or the phase reference are undetermined. This is why perturbations of the steady-state oscillation in the direction of the limit cycle can increase arbitrarily, which gives rise to the phase noise. Taking this into account, in [1]–[4], the differential equations of the noisy oscillator are linearized and solved for the transversal and tangential components of the perturbation to the limit cycle. The tangential component provides the phase noise. In [1]–[4], this component is obtained from the linearized time-periodic differential equation, requiring the computation of the time-domain Jacobian matrix. A different time-domain approach is the one based on the calculation of the impulse-sensitivity function (ISF) [5], which gives a measure of the phase-excess response of the linearized oscillator (about the steady-state regime) to a given noise source. As in [1]–[4], the calculation of this function also makes use of the tangential projection of the perturbation over the limit cycle. The authors propose an approximate analytical expression for the ISF, providing this projection [5]. To distinguish the analytical expression of the ISF from its (more accurate) numerical calculation, the former will be denoted here as ISF based on tangential perturbation (TP-ISF). Unlike the calculations in [1]–[4], it does not require the system time-domain Jacobian matrix. The time-derivative of the oscillator steady-state solution is used instead [5]. The TP-ISF has recently been extended to the calculation of phase noise from commercial harmonic balance [6]. When using harmonic balance, one of the most common approaches is the one based on the computation of the conversion matrix [7]–[10]. For small frequency offsets from the carrier, [7] makes use of the *mixed-mode* harmonic-balance formulation, to take into account the frequency modulation of the carrier.

The initial objective of this study has been the development of a technique enabling the application of the rigorous stochastic characterization of phase noise, based on Floquet's analysis, to oscillators simulated through harmonic balance. The application of harmonic balance in conjunction with Floquet's analysis is briefly described in [3]. The aim here has been to establish the relationship with general and practical harmonic-balance formulations [6], [7]. This initial objective has led us to an in-depth analytical study of three different techniques, i.e., the one based on Floquet's analysis, the one based on the TP-ISF, and the carrier-modulation approach. The analytical comparison has provided clear relationships between these three approaches. These relationships were thought to be of interest for the microwave designer and are presented here. Comparisons between the carrier modulation and conversion-matrix approach will not be carried out since this has already been done in [7] in a very rigorous way.

The paper is organized as follows. In Section I, the formulation [1]–[4] is compared with the one based on the ISF [5]. In Section II, a frequency-domain approach to the differential equation, describing the perturbed oscillator [1], [2], is presented, enabling a comparison with the carrier-modulation analysis. Sensitivity functions to noise perturbations are calculated, allowing the application of the phase-noise stochastic characterization from Floquet's analysis to circuits simulated with harmonic balance. In Section III, the application of the phase-noise analysis to a cubic nonlinearity oscillator allows numerical calculations and accuracy comparisons. Note that, due the nature of this paper, consisting of an analytical comparison between different techniques, some of the expressions provided by their corresponding authors had to be rewritten for the sake of clarity.

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II. Comparison of Time-Domain Analysis Techniques

The nonlinear differential equation ruling the free-running oscillator, in the presence of noise sources, is given by

$$\dot{\overline{x}} = \overline{f}\left(\overline{x}, \overline{\xi}, \gamma_1, \dots, \gamma_m\right), \qquad \overline{x} \in R^n; \quad \overline{f} \in R^n; \quad \overline{\xi} \in R^s, \gamma_1, \dots, \gamma_m \in R \tag{1a}$$

$$\langle \xi_i(t)\xi_j(t)\rangle = \Gamma_{ij}\delta(t-t') \tag{1b}$$

with $\overline{x}$, $\overline{f}$, and $\overline{\xi}$, respectively, being the state variables, the vector of nonlinear functions, and the vector of white-noise sources, and γ_i, with $i = 1$ to m, being the colored noise sources. The matrix $[\Gamma]$ has dimension $s \times s$ and provides the correlation of the white-noise sources. In [1]–[4], (1) is linearized about the steady-state limit cycle, considering two components, i.e., the stochastic time shift θ [since $\overline{x} = \overline{x}(t+\theta)$] and the transversal amplitude variation $\Delta\overline{x}$. After elimination of second-order terms, the linearized equation is as follows [1], [2]:

$$\overline{x}'_o(y)\dot{\theta} + \Delta\overline{x}'(y) = \left[Df\left(\overline{x}_o(y)\right)\right]\Delta\overline{x}(y) + \left[g_w\left(\overline{x}_o(y)\right)\right]\overline{\xi}(t) + \sum_{i=1}^{m}\left[g_i^{\gamma}\left(\overline{x}_o(y)\right)\right]\gamma_i(t) \tag{2}$$

where $\overline{x}_o(t)$ is the steady-state solution, y is the perturbed time, $y = t+\theta$, $[Df]$ is the Jacobian matrix of $\overline{f}$ with respect to the state variables, $[g_w(y)]$ is the Jacobian matrix of $\overline{f}$ with respect to the white noise sources $\overline{\xi}$, and each $[g_i^{\gamma}(y)]$ is a column matrix, containing the derivatives of $\overline{f}$ with respect to the colored noise source i. The two unknowns of (2) are $\Delta\overline{x}$ and θ. The equation is linear in $\Delta\overline{x}$, but nonlinear in θ.

In [1], in order to solve (2) for θ and $\Delta\overline{x}$, the equation is alternatively multiplied by a projection vector $\overline{n}(t)$ tangent to the cycle and by a projection operator orthogonal to the cycle $[P(t)] = [I] - \overline{n}(t)\cdot\overline{n}(t)^T$. The resulting equation for the stochastic time-shift is [1]

$$\dot{\theta}(t) = [\beta(y)]\Delta\overline{x}(y) + \frac{\overline{n}(y)^T\left[g_w\left(x_o(y)\right)\right]}{\left|\overline{x}'_o(y)\right|}\overline{\xi}(t) + \frac{\overline{n}(y)^T}{\left|\overline{x}'_o(y)\right|}\sum_{i=1}^{m}\left[g_i^{\gamma}\left(x_o(y)\right)\right]\gamma_i(t), \quad \text{with } \overline{n}(t) = \frac{\dot{\overline{x}}_o(t)}{\left|\dot{\overline{x}}_o(t)\right|}. \tag{3}$$

The matrix $[\beta]$ exclusively depends on the system time-domain Jacobian matrix and the projection elements $\overline{n}$ and $[P]$. In [2], a different resolution of (2) is presented, in which the equations for $\Delta\overline{x}$ and $\theta(t)$ are uncoupled using the Floquet vector $\overline{v}_1^T(t)$ of the adjoint linearized equations [2] associated with the Floquet multiplier 1. The stochastic time shift is now given by

$$\dot{\theta}(t) = \overline{v}_1^T(y)\left[g_w\left(\overline{x}_o(y)\right)\right]\overline{\xi}(t) + \overline{v}_1^T(t)\sum_{i=1}^{m}\left[g_i^{\gamma}\left(\overline{x}_o(y)\right)\right]\gamma_i(t). \tag{4}$$

Note that (4) is a nonlinear equation in θ [in the same way as (3)] since $y = t+\theta$. Thus, (4) cannot be solved through direct integration. However, the correlation spectrum of the phase noise (obtained from [4]) only depends on the periodic vector $\overline{c}_w^T(t) = [c_{w1}(t) \ \cdots \ c_{ws}(t)] \equiv \overline{v}_1^T(t)[g_w(\overline{x}_o(t))]$ and the scalars $c_{\gamma i}(t) \equiv \overline{v}_1^T(t)[g_i^{\gamma}(\overline{x}_o(t))]$. These functions provide the sensitivity of the oscillator phase to the noise sources (white or colored). As an example, for a set of white noise sources, with correlation matrix $[\Gamma]$ and a colored-noise source, with spectral density $S_F(f_m)$, uncorrelated with the white-noise sources, the phase-noise spectrum about the carrier frequency f_o is approached by [4]

$$S_{\phi}(f_m) = f_o^2\frac{CW}{f_m^2} + f_o^2\left|C_{\gamma}^o\right|^2\frac{S_F(f_m)}{f_m^2} \tag{5}$$

where f_m is the frequency offset from the carrier, $CW = (1/T)\int_0^T \overline{c}_w^T(t)[\Gamma]\overline{c}_w(t)\,dt$, and C_{γ}^o is the dc component of $c_{\gamma}(t)$. The extension to multiple white and colored noise sources is immediate and has been carried out in [4].

In [5], a different time-domain technique is proposed. It is based on the calculation of the linear-time variant impulsive response of the linearized oscillator (about its steady-state regime). This impulsive response provides the phase shift of the linearized oscillator versus the input current or voltage, and is given by $h(t,\tau) = \Gamma(\tau)u(t-\tau)$. The function Γ is called ISF [4]. For a current noise source $i(t)$ (either white or colored), the time shift [related to the phase shift through $\phi(t) = \omega_o\theta(t)$] is calculated through the superposition integral $\theta(t) = (1/\omega_o)\int_{-\infty}^{t}\Gamma(\omega_o\tau)i(\tau)\,d\tau$. As will be shown, this linear-time variant calculation is equivalent to the direct integration of the time shift in (3).

The authors propose two main techniques for the ISF calculation. One is a numerical time-domain technique. The other is an approximate analytical expression [5]. This approximate expression provides the tangential projection of the perturbation over the limit cycle and, as already indicated, is referred to here as the TP-ISF. For the case of a current noise generator $i(t)$, connected at the circuit node j, the analytical expression of the ISF is given by

$$\Gamma(\omega_o\tau) = \omega_o\frac{1}{C_j}\cdot\frac{\dot{x}_{oj}(\tau)}{\left|\dot{\overline{x}}_o(\tau)\right|^2} \tag{6}$$

where C_j is the equivalent capacitance at the particular node j. A dual formulation exists for the case of a voltage noise source [5]. From (6), the time shift is given by

$$\theta(t) = \int_{-\infty}^{t}\frac{\dot{x}_{oj}(\tau)}{\left|\dot{\overline{x}}_o(\tau)\right|^2}\frac{1}{C_j}i(\tau)\,d\tau. \tag{7}$$

The time in the periodic term inside the integral should actually be shifted with respect to the noise source, i.e.,

$$\frac{\dot{x}_{oj}(\tau+\theta)}{\left|\dot{\overline{x}}_o(\tau+\theta)\right|^2}.$$

Again, although the calculation of the time shift θ based on the impulsive response $h(t,\tau)$ is incorrect (due to the dependence of the integrand on θ [3]), the statistical properties of the phase shift only depend on the periodic function $\Gamma(\omega_o\tau)$ [5].

Let (3) and (7) be compared now. In case of an oscillator circuit, with a current noise source $i(t)$ in parallel with a capacitance (the case considered in [5]), the dominant term of the derivative of $\Delta\dot{v}_j$ with respect to the current noise source $i(t)$ is $1/C_j$. Thus, $g(y) = 1/C_j$ and the analytical expression (7) is a particular case of (3), for a parallel oscillator topology, with a current noise source. Note that the factor

$$\frac{\dot{x}_{oj}(\tau)}{C_j\left|\dot{\overline{x}}_o(\tau)\right|^2}$$

in (7) agrees with the term

$$\frac{\overline{n}(y)^T\left[g(y)\right]}{\left|\overline{x}'_o(y)\right|}$$

in (3) for only one noise source $i(t)$, with $g(y)$ being the Jacobian term affecting the particular (white or colored) noise source. However, the first term of (3), providing the influence of the amplitude perturbation on the phase shift, does not appear in (7). Thus, (7) is an approximation to (3).

In fact, the multiplication by the factor

$$\frac{\dot{x}_{oj}(t)}{\left|\dot{\overline{x}}_o(t)\right|^2}$$

provides an accurate projection over the cycle, but it does not uncouple the phase and amplitude perturbations, as the multiplication by the vector $\overline{v}_1^T$ does. In (3), the phase shift is not solely provided by the projection over the cycle of the noise-source perturbation. There is a transient along which $\Delta\overline{x}$ evolves and this state-variable perturbation must also be projected over the cycle. Due to limiting effects inherent to the nonlinearity of the oscillator, the projection of $\Delta\overline{x}$ will usually have smaller influence on the phase-shift value.

A problem in the ISF formulation comes from the fact that, in the way that it has been presented in [5], it is restricted to only two possible kinds of noise source models, i.e., a current source in parallel with a capacitance or a voltage source in series with an inductor. However, the designer is often provided with models that do not match any of those two situations. In order to avoid this restriction, a more general analytical expression (but also neglecting the contribution of $\Delta\overline{x}$) is proposed here, based on the comparison between (3) and (7) as follows:

$$\Gamma_i(\omega_o t) = \omega_o\frac{\overline{n}(y)^T\left[g(y)\right]}{\left|\overline{x}'_o(y)\right|} \tag{8}$$

where $[g(y)]$ is the column matrix containing the derivatives of the nonlinear function $\overline{f}$ with respect to the particular noise source. As in the case of (4) and (5), the calculation of the phase-noise spectrum only depends on the Fourier coefficients of the former periodic function.

III. Frequency-Domain Analysis Techniques

A. Perturbed-Oscillator Equations in the Frequency Domain

For the frequency-domain approach to (2), the stochastic time shift can be written as

$$\theta(t) = \int_0^t \frac{\Delta\omega(\lambda)}{\omega_o}\,d\lambda. \tag{9}$$

Assuming slowly varying perturbations, $\Delta\omega(t)$ can be expressed in the form

$$\Delta\omega(t) = \sum_k \Delta\omega_k(t)e^{jk\omega_o t} \tag{10}$$

with $\Delta\omega_k(t)$ being slowly varying terms. In the following, and under the assumption of slowly varying perturbations, only the term with zero index $\Delta\omega_o(t)$ will be considered. Using the Fourier-series expansion of $\overline{x}_o(t)$, it is then possible to write the two first terms of (2) in the form

$$\begin{aligned}\overline{x}'_o(y)\dot{\theta} &\cong \sum_k \overline{X}^k jk\Delta\omega_o e^{jk\omega_o y}\\ &= \sum_k \overline{X}^k jk\Delta\omega_o e^{jk\omega_o\theta}e^{jk\omega_o t}\\ \Delta\overline{x}'(y) &= \sum_k\left(\Delta\overline{X}^k(y)jk\omega_o + \Delta\dot{\overline{X}}^k(y)\right)e^{jk\omega_o\theta}e^{jk\omega_o t}.\end{aligned} \tag{11}$$

Replacing (11) into (2) and equating the terms that correspond to the same harmonic order, the following expression is obtained:

$$\begin{bmatrix} -jNH &&&&&& \\ & \ddots &&&&& \\ && jNH &&&& \\ &&& \ddots &&& \\ &&&& -jNH && \\ &&&&& \ddots & \\ &&&&&& jNH \end{bmatrix}\begin{bmatrix} X_1^{-NH} \\ \vdots \\ X_1^{NH} \\ \vdots \\ X_n^{-NH} \\ \vdots \\ X_n^{NH} \end{bmatrix}\Delta\omega_o$$

$$+\begin{bmatrix} -jNH &&&&&& \\ & \ddots &&&&& \\ && jNH &&&& \\ &&& \ddots &&& \\ &&&& -jNH && \\ &&&&& \ddots & \\ &&&&&& jNH \end{bmatrix}$$

$$\bullet\begin{bmatrix} \Delta X_1^{-NH} \\ \vdots \\ \Delta X_1^{NH} \\ \vdots \\ \Delta X_n^{-NH} \\ \vdots \\ \Delta X_n^{NH} \end{bmatrix}\omega_o + \begin{bmatrix} \Delta\dot{X}_1^{-NH} \\ \vdots \\ \Delta\dot{X}_1^{NH} \\ \vdots \\ \Delta\dot{X}_n^{-NH} \\ \vdots \\ \Delta\dot{X}_n^{NH} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial F_1^{-NH}}{\partial X_1^{-NH}} & \cdots & \frac{\partial F_1^{-NH}}{\partial X_1^{NH}} & \cdots & \frac{\partial F_1^{-NH}}{\partial X_n^{-NH}} & \cdots & \frac{\partial F_1^{-NH}}{\partial X_n^{NH}} \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \frac{\partial F_1^{NH}}{\partial X_1^{-NH}} & \cdots & \frac{\partial F_1^{NH}}{\partial X_1^{NH}} & \cdots & \frac{\partial F_1^{NH}}{\partial X_n^{-NH}} & \cdots & \frac{\partial F_1^{NH}}{\partial X_n^{NH}} \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \frac{\partial F_n^{-NH}}{\partial X_1^{-NH}} & \cdots & \frac{\partial F_n^{-NH}}{\partial X_1^{NH}} & \cdots & \frac{\partial F_n^{-NH}}{\partial X_n^{-NH}} & \cdots & \frac{\partial F_n^{-NH}}{\partial X_n^{NH}} \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \frac{\partial F_n^{NH}}{\partial X_1^{-NH}} & \cdots & \frac{\partial F_n^{NH}}{\partial X_1^{NH}} & \cdots & \frac{\partial F_n^{NH}}{\partial X_n^{-NH}} & \cdots & \frac{\partial F_n^{NH}}{\partial X_n^{NH}} \end{bmatrix}$$

$$
\bullet \begin{bmatrix} \Delta X_1^{-NH} \\ \vdots \\ \Delta X_1^{NH} \\ \vdots \\ \Delta X_n^{-NH} \\ \vdots \\ \Delta X_n^{NH} \end{bmatrix}
+ \begin{bmatrix}
\frac{\partial F_1^{-NH}}{\partial E_1^{-NH}} & \cdots & \frac{\partial F_1^{-NH}}{\partial E_1^{NH}} & \cdots & \frac{\partial F_1^{-NH}}{\partial E_{s+m}^{-NH}} & \cdots & \frac{\partial F_1^{-NH}}{\partial E_{s+m}^{NH}} \\
\vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
\frac{\partial F_1^{NH}}{\partial E_1^{-NH}} & \cdots & \frac{\partial F_1^{NH}}{\partial E_1^{NH}} & \cdots & \frac{\partial F_1^{NH}}{\partial E_{s+m}^{-NH}} & \cdots & \frac{\partial F_1^{NH}}{\partial E_{s+m}^{NH}} \\
\vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
\frac{\partial F_n^{-NH}}{\partial E_1^{-NH}} & \cdots & \frac{\partial F_n^{-NH}}{\partial E_1^{NH}} & \cdots & \frac{\partial F_n^{-NH}}{\partial E_{s+m}^{-NH}} & \cdots & \frac{\partial F_n^{-NH}}{\partial E_{s+m}^{NH}} \\
\vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
\frac{\partial F_n^{NH}}{\partial E_1^{-NH}} & \cdots & \frac{\partial F_n^{NH}}{\partial E_1^{NH}} & \cdots & \frac{\partial F_n^{NH}}{\partial E_{s+m}^{-NH}} & \cdots & \frac{\partial F_n^{NH}}{\partial E_{s+m}^{NH}}
\end{bmatrix}
\bullet \begin{bmatrix}
e^{jNH\omega_o\theta} & & & & & \\
& \ddots & & & & \\
& & e^{-jNH\omega_o\theta} & & & \\
& & & \ddots & & \\
& & & & e^{jNH\omega_o\theta} & \\
& & & & & \ddots \\
& & & & & & e^{-jNH\omega_o\theta}
\end{bmatrix}
\bullet \begin{bmatrix} E_1^{-NH}(t) \\ \vdots \\ E_1^{NH}(t) \\ \vdots \\ E_{s+m}^{-NH}(t) \\ \vdots \\ E_{s+m}^{NH}(t) \end{bmatrix} \qquad (12)
$$

where n is the number of state variables, NH is the number of harmonic components, and $E(t)$ is the vector, with dimension $(s+m)(2NH+1)$, containing the harmonic terms of the slowly varying noise sources. Note that s white noise generators plus m colored generators have been taken into account. The matrices $[\partial\overline{F}/\partial\overline{X}]$ agrees with the Toeplitz matrix [12] (or the frequency-domain expression) of $[Df]$, and $[\partial\overline{F}/\partial\overline{E}]$, with the Toeplitz matrix of the Jacobian with respect to the noise sources, including both $[g_w]$ and $[g_i^\gamma]$. The matrix containing the exponential terms $e^{-jk\omega_o\theta}$ comes from the fact that all the harmonic terms, except the noise sources, are referred to the temporal variable y. The terms $e^{-jk\omega_o\theta}$ account for this time shift. For simplicity, (12) will be rewritten in a compact way as

$$[jk]\Delta\omega_o\overline{X}+[jk]\omega_o\Delta\overline{X}+\Delta\dot{\overline{X}} = [DF]\Delta\overline{X}+[G][e^{-jk\omega_o\theta}]\overline{E}(t) \qquad (13)$$

where $[jk]$ and $[e^{-jk\omega_o\theta}]$ are the diagonal matrices in (12), with respective dimension $n(2NH+1)\times n(2NH+1)$ and $(s+m)(2NH+1)\times(s+m)(2NH+1)$. On the other hand, $[DF] = [\partial\overline{F}/\partial\overline{X}]$ and $[G] = [\partial\overline{F}/\partial\overline{E}]$. In an analogous way to (4), (13) is nonlinear in θ.

B. Harmonic-Balance Formulation

The harmonic equation (13) can be written in a more compact and useful way in terms of the harmonic-balance error function. The general harmonic-balance formulation uses Jacobian matrices of this error function. To obtain this equivalent formulation, (1), in the absence of noise sources, can be rewritten by introducing the auxiliary error function $h(x)$: $R^n \rightarrow R^n$ as follows:

$$\overline{h}(x) = \dot{\overline{x}} - \overline{f}\,(\overline{x}) = \overline{0} \qquad (14)$$

which, for the steady-state oscillation, can be translated to the frequency domain as

$$\overline{H}\left(\overline{X}_o, \omega_o\right) = [jk]\omega_o\overline{X}_o - \overline{F}\left(\overline{X}_o\right) = 0. \qquad (15)$$

Note that the matrix $[jk]$ is the same as the one in (12). In the presence of noise sources, the linear expansion of the harmonic-balance equation around the steady state, eliminating the steady-state terms, can be written as follows:

$$\begin{aligned}\Delta\overline{H} &= [jk]\omega_o\Delta\overline{X} - [DF]\Delta\overline{X} + [jk]\Delta\omega_o\overline{X}_o \\ &\quad + \Delta\dot{\overline{X}} - \left.\frac{\partial\overline{F}}{\partial\overline{E}}\right|_o [e^{-jk\omega_o\theta}]\overline{E} \\ &= 0. \end{aligned} \qquad (16)$$

Assembling terms, the following relationships can be introduced:

$$\begin{aligned} [jk]\omega_o - [DF] &= \left.\frac{\partial\overline{H}}{\partial\overline{X}}\right|_o \equiv [JH] \\ [jk]\overline{X}_o &\equiv \left.\frac{\partial\overline{H}}{\partial\omega}\right|_o \\ [G] &\equiv -\left[\frac{\partial\overline{F}}{\partial\overline{E}}\right]_o = \left[\frac{\partial\overline{H}}{\partial\overline{E}}\right]_o \end{aligned} \qquad (17)$$

and (13) becomes

$$\left.\frac{\partial\overline{H}}{\partial\omega}\right|_o \Delta\omega_o(t)+[JH]_o\Delta\overline{X}(t)+\Delta\dot{\overline{X}}(t) = [G]\left[e^{-jk\omega_o\theta}\right]\overline{E}(t). \qquad (18)$$

Due to the oscillator autonomy, the Jacobian $[JH]$ is a singular matrix. Let the vector $\overline{V}_1$ belong to the kernel of the adjoint matrix $[JH]_o^+$. This vector can be chosen so as its adjoint vector fulfills $\overline{V}_1^+\cdot(\partial\overline{H}/\partial\omega)|_o = 1$. Since there is one degree of freedom in $\Delta\overline{X}$, it can also be imposed, i.e., $\overline{V}_1^+\cdot\Delta\dot{\overline{X}} = 0$. The system (18) can then be uncoupled through its left-hand-side multiplication by the vector $\overline{V}_1^+$ (providing a system in $\Delta\omega$) or by the auxiliary matrix $[M] = [I] - (\partial\overline{H}/\partial\omega)|_o \cdot \overline{V}_1^+$ (providing, together with $\overline{V}_1^+ \cdot \Delta\overline{X} = \overline{0}$, a system in $\Delta\overline{X}$). The left-hand-side multiplication of (13) by $\overline{V}_1^+$ gives rise to

$$\Delta\omega_o(t) = \overline{V}_1^+ [G]\Big[e^{-jk\omega_o\theta}\Big]\,\overline{E}(t) \tag{19}$$

where the normalization $\overline{V}_1^+ \cdot (\partial\overline{H}/\partial\omega)|_o = 1$ has been taken into account. This equation is equivalent to (4).

In case of a general harmonic balance formulation (with a typically smaller number [11] of state variables and linear matrices of higher order in ω_o), the expression for $\Delta\omega(t)$ is also given by (19), as shown in the Appendix. Due to the relationship between $\Delta\omega(t)$ and $\theta(t)$ in (9), (19) is, in fact, a nonlinear equation in $\Delta\omega_o(t)$, in agreement with (4). Thus, a nonlinear resolution of (19) would be required to determine $\Delta\omega_o(t)$ for a given realization of $\overline{E}(t)$. However, the objective is the calculation of the phase-noise spectral density [7] in terms of the spectral densities of the noise sources. The multiplication by the complex conjugate of the coefficient affecting $\overline{E}(t)$ then makes the dependence on θ vanish in the common case of uncorrelated noise sources. Equation (19) shows the close similarity between the time-domain calculation and the carrier-modulation approach [7]. The only difference between (19) and [7] is the actual resolution of the perturbed-oscillator equation. In the mixed-mode approach [7], the singularity of the harmonic-balance equations is removed by arbitrarily imposing $\mathrm{Imag}(\Delta X_k^i) = 0$ for the harmonic component k of one of the variables i. In (19), the condition $\overline{V}_1^+[JH] = \overline{0}$ has been used. It can easily be shown that the result, in terms of $\Delta\omega_o$, is exactly the same, thus, (19) is a different expression of the carrier modulation approach.

Comparison in now carried out with the phase-noise calculation in [8]. In [8], the perturbed solution is *a priori* expanded in the sidebands $k\omega_o + \omega_m$, as in the conversion-matrix approach. Since the sidebands are prefixed, the Jacobian matrix $[JH(k\omega_o + \omega_m)]$ is no longer singular. The authors use the vector $\overline{V}_1$, but the purpose is not to eliminate the singular term $[JH]$, but to solve the numerical problems arising in the conversion-matrix technique for small values of the frequency offset ω_m. They perform a Taylor-series development of first order of the Jacobian matrix $[JH(k\omega_o + \omega_m)]$ in ω_m and employ an eigenvalue and eigenvector calculation (involving $\overline{V}_1$) for overcoming the matrix-inversion problem near the carrier.

IV. Sensitivity Functions to Noise Perturbations

From the time-domain equation (4), and taking (9) into account, it is possible to write

$$\begin{aligned}\Delta\omega(t) &= \omega_o\overline{v}_1^T(y)\Big[g_w\big(\overline{x}_o(y)\big)\Big]\overline{\xi}(t)\\ &\quad+\omega_o\overline{v}_1^T(y)\sum_{i=1}^{m}\Big[g_i^\gamma\big(\overline{x}_o(y)\big)\Big]\gamma_i(t)\\ &=\omega_o\overline{c}_w^T(y)\overline{\xi}(t)+\omega_o\sum_{i=1}^{m}c_i^\gamma(y)\gamma_i(t)\end{aligned} \tag{20}$$

where the introduced terms $\overline{c}_w(y)$ and $c_i^\gamma(y)$ were already defined in Section I. To express (20) in the frequency domain, it is necessary to make use of the Toeplitz matrices [12] associated to $\overline{c}_w(y)$ and $c_i^\gamma(y)$, respectively, called $[TC_w]$ and $[TC_i^\gamma]$. The following expression is obtained:

$$\begin{aligned}[\Delta\omega(t)] &= \omega_o[TC_w]\Big[e^{-jk\omega_o\theta}\Big]_w\overline{E}_w(t)\\ &\quad+\omega_o\sum_{i=1}^{m}\Big[TC_i^\gamma\Big]\Big[e^{-jk\omega_o\theta}\Big]_\gamma\overline{E}_i^\gamma(t)\end{aligned} \tag{21}$$

where $[\Delta\omega(t)]$ is a column matrix, containing the harmonic terms of the expansion (10), $\overline{E}_w(t)$ is an harmonic vector, containing the white-noise sources, and $\overline{E}_i^\gamma(t)$ refers to each of the colored sources. If only the zero-index term is considered in $[\Delta\omega(t)]$, (21) simplifies to

$$\begin{aligned}\Delta\omega_o(t) &= \omega_o\overline{C}_w^+\Big[e^{-jk\omega_o\theta}\Big]_w\overline{E}_w(t)\\ &\quad+\omega_o\sum_{i=1}^{m}\overline{C}_{\gamma i}^+\Big[e^{-jk\omega_o\theta}\Big]_\gamma\overline{E}_i^\gamma(t)\end{aligned} \tag{22}$$

where $\overline{C}_w$ and $\overline{C}_{\gamma i}$, respectively, contain the harmonic terms of the periodic functions $\overline{c}_w(t)$ and $c_i^\gamma(t)$ in suitable order. To compare (22) with (19), the latter equation can be decomposed according to

$$\begin{aligned}\Delta\omega_o(t) &= \overline{V}_1^+[G]\Big[e^{-jk\omega_o\theta}\Big]\overline{E}(t)\\ &= \overline{V}_1^+[G_w]\Big[e^{-jk\omega_o\theta}\Big]_w\overline{E}_w(t)\\ &\quad+\sum_{i=1}^{m}\overline{V}_1^+[G_i^\gamma]\Big[e^{-jk\omega_o\theta}\Big]_\gamma\overline{E}_i^\gamma(t)\end{aligned} \tag{23}$$

where $[G_w]$ refers to the white-noise sources and $[G_i^\gamma]$ refers to each of the colored sources. Now comparing (23) with (22)

$$\begin{aligned}\omega_o\overline{C}_w^+ &= \overline{V}_1^+[G_w] = \overline{B}_w^+\\ \omega_o\overline{C}_{\gamma i}^+ &= \overline{V}_1^+[G_i^\gamma] = \overline{B}_{\gamma i}^+, \qquad \text{with } i = 1,\ldots,m\end{aligned} \tag{24}$$

where $\overline{B}_w = [B_1^{-NH}\cdots B_1^{NH}\cdots B_s^{-NH}\cdots B_s^{NH}]$ and $\overline{B}_{\gamma i} = [B_{\gamma i}^{-NH}\cdots B_{\gamma i}^{NH}]$. The term B_i^k refers to the harmonic k affecting the white noise source i (with $i = 1,\ldots,s$) and $B_{\gamma i}^k$ refers to the harmonic k affecting the colored noise source i. Assembling the harmonic terms that refer to the same white noise source, it is possible to obtain the following time-domain matrix:

$$\begin{aligned}&\big[b_{w1}(t)\quad\cdots\quad b_{ws}(t)\big]\\ &\quad=\left[\sum_{k=-NH}^{NH}B_1^k e^{jk\omega_o t}\cdots\sum_{k=-NH}^{NH}B_s^k e^{jk\omega_o t}\right]\end{aligned} \tag{25}$$

and for the terms referring to the colored noise sources

$$b_i^\gamma(t) = \sum_{k=-NH}^{NH}B_{\gamma i}^k e^{jk\omega_o t}, \qquad i = 1,\ldots,m. \tag{26}$$

The following relationships are then fulfilled:

$$\begin{aligned}c_{wi}(t) &= \frac{1}{\omega_o}\,b_{wi}(t), \qquad \text{with } i = 1,\ldots,s\\ c_i^\gamma(t) &= \frac{1}{\omega_o}\,b_i^\gamma(t), \qquad \text{with } i = 1,\ldots,m.\end{aligned} \tag{27}$$

These periodic coefficients provide the oscillator phase sensitivity to the noise sources. They are used in [1]–[4] for the stochastic characterization of the phase-noise spectrum. This spectrum [1]–[4] depends on the constant term, referring to the white noise sources $CW = (1/T)\int_0^T \bar{c}_w^T(t)[\Gamma]\bar{c}_w(t)\,dt = BW/\omega_o^2$ and the dc terms $C_{\gamma i}^o$, whose calculation is straightforward from the equivalencies (27).

Sensitivity functions (25) and (26) enable the application of all the stochastic characterization of phase noise, carried out in [4] to oscillators simulated through harmonic balance. This analysis is very general, enabling the calculation of different limiting forms of the noisy oscillator spectrum, for different frequency ranges of interest. In the expressions provided in [4], it is possible to take into account a realistic model of the $1/f$ noise. In [2], the $1/f$ noise is modeled with an infinite sum of autocorrelation spectra of Ornstein–Uhlenbeck processes, which are statistically independent and have damping rates γ ranging from zero to infinite. For damping rate tending to zero, the correlation time of the process tends to infinite. According to [13], the singularity of the spectral density at $f = 0$ arises as a result of the actual nonstationary nature of this stochastic process. In order to avoid the nonphysical situation, a cutoff frequency $f_{\min}$ is introduced in [4]. The expression modeling the $1/f$ noise [4, eq. (66)] is $S_\gamma(f) = 1/|f| - 4/(2\pi f)\arctan(f_{\min}/(2\pi f))$. The $1/f$ noise then behaves as a stationary process. This cutoff frequency can be related to the finite measurement time T as $f_{\min} = 1/T$ [13]. The frequency $f_{\min}$ will typically be very small. The variance of the time deviation $\theta(t)$ is then approached (see [4, eq. (59)]) by $\sigma^2(t) = 2|C_\gamma^o|^2 \int_{-\infty}^{\infty} S_\gamma(f)(1 - e^{j2\pi ft})/(4\pi^2 f^2)\,df$. For the calculation of the oscillator spectrum due to phase noise, the Fourier transform of the autocovariance function of $\bar{x}_o(t+\theta(t))$ is then determined [4] through approximating series expansions [14]. Expressions (25) and (26) make this accurate spectrum calculation applicable from the harmonic-balance simulation of the oscillator circuit. This is done by combining (25) and (26) with the expression provided in [4] for the oscillator spectrum, which is exclusively due to phase noise. For m Gaussian $1/f$ noise sources, uncorrelated with each other and with the s white-noise sources, the spectrum, at each harmonic k, is given by

$$S_k(f_m) = \begin{cases} \dfrac{\dfrac{k^2}{4\pi^2}\left(BW + \sum\limits_{i=1}^{m} \left|B_{\gamma i}^o\right|^2 S_{\gamma i}(0)\right)}{\dfrac{k^4}{16\pi^2}\left(BW + \sum\limits_{i=1}^{m} \left|B_{\gamma i}^o\right|^2 S_{\gamma i}(0)\right)^2 + f_m^2}, & f_m \approx 0 \quad \text{(28a)} \\ \dfrac{k^2}{4\pi^2}\left(BW + \sum\limits_{i=1}^{m} \left|B_{\gamma i}^o\right|^2 S_{\gamma i}(f_m)\right)\dfrac{1}{f_m^2}, & f_m \gg 0 \quad \text{(28b)} \end{cases}$$

with $BW = (1/T)\int_0^T \bar{b}_w^T(t)[\Gamma]\bar{b}_w(t)\,dt$ and $S_{\gamma i}(f_m)$ being the spectral density of the colored noise source $\gamma_i(t)$, with $i = 1$ to m. Reference [2] provides the asymptotic expression of the oscillator phase-noise spectrum for offset frequencies $f_m \gg f_{\min}$ (see [2, eq. (141)]), and this expression is found to be independent of the measurement time T. The expression coincides with (28b) and with the phase-noise spectrum obtained through the carrier-modulation approach in [7].

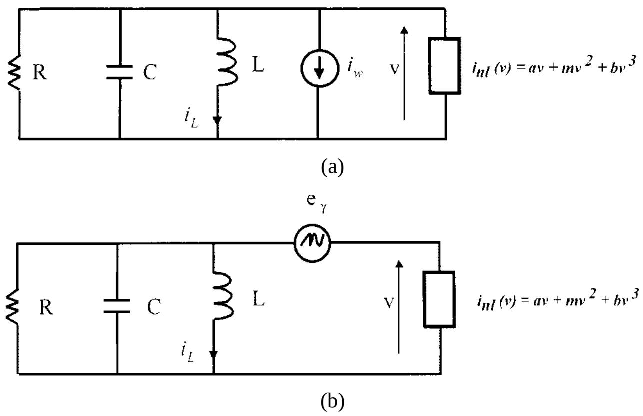


Fig. 1. Schematic of the cubic nonlinearity oscillator. The parameters of the nonlinear element are $a = -0.037$ A/V, $m = 0.01$ A/V^2, and $b = 0.021$ A/V^3. The values of the circuit elements are $R = 45.5871\ \Omega$, $L = 10$ nH, and $C = 2.0651$ pF. The free-running oscillation frequency is $\omega_o = 2\pi 10^9$ s^{-1}. (a) Inclusion of a current noise source. (b) Inclusion of a voltage noise source.

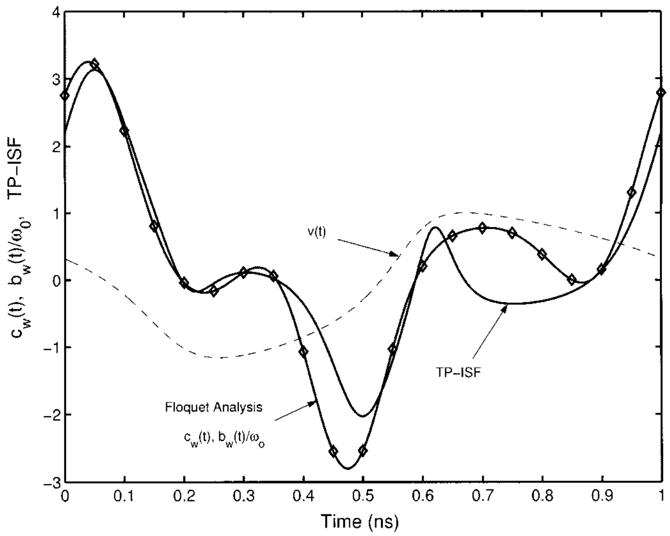
As can be gathered from (28), the minimization of the dc value of the sensitivity functions $b_i^\gamma(t)$ minimizes the up-conversion of $1/f$ noise. On the other hand, the minimization of BW minimizes the influence of white noise. The use of noise-sensitivity functions has opened new possibilities for low phase-noise oscillator design [5]–[15]. In [15], for instance, some design decisions for ring oscillators (the type of implementation and the number of stages) are taken in terms of the rms and dc values of the ISF. The phase-noise reduction is verified in the experiment.

V. Numerical Applications

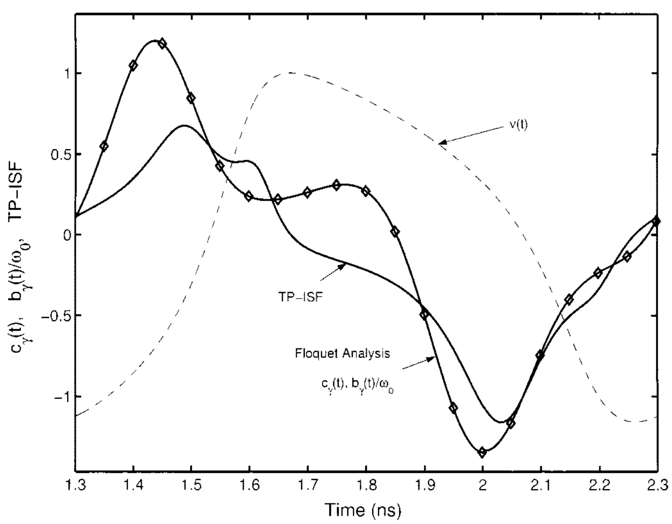
The phase noise of the cubic nonlinearity oscillator of Fig. 1 has been calculated using the three different techniques that were analytically compared in the previous sections. The parameters of the nonlinearity are $a = -0.037$ A/V, $m = 0.01$ A/V^2, and $b = 0.021$ A/V^3. The values of the circuit elements are $R = 45.5871\ \Omega$, $L = 10$ nH, and $C = 2.0651$ pF. The free-running oscillation frequency is $\omega_o = 2\pi 10^9$ s^{-1}. Two different noise models have been considered, i.e., one with a current noise source, in parallel with the nonlinearity [see Fig. 1(a)], and the other with a voltage noise source in series with the nonlinearity [see Fig. 1(b)]. The first model does not provide up-conversion of $1/f$ noise [4].

For a white-noise current source $i_w(t)$, in parallel across the nonlinearity, the perturbed nonlinear differential equation is given by [see (2)]

$$\begin{bmatrix} \dfrac{di_{Lo}}{dy} \\ \dfrac{dv_o}{dy} \end{bmatrix} \dot{\theta} + \begin{bmatrix} \dfrac{d\Delta i_L}{dy} \\ \dfrac{d\Delta v}{dy} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{L} \\ \dfrac{-1}{C} & -\dfrac{p}{C} - \dfrac{2m}{C} v_o(y) - \dfrac{3bv_o^2(y)}{C} \end{bmatrix} \begin{bmatrix} \Delta i_L \\ \Delta v \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{-1}{C} \end{bmatrix} \cdot i_w(t), \qquad \text{with } p = a + \frac{1}{R}. \tag{29}$$



(a)



(b)

Fig. 2. Cubic-nonlinearity oscillator. Comparison between $c_w(t)$ and $c_\gamma(t)$, calculated through Floquet's theory and through the frequency-domain technique (diamonds), and the ISF, based on tangential perturbation. The calculation has been carried out with state variables normalized to their maximum value. The waveform $v(t)$ is also superimposed. (a) Voltage noise source. (b) Current noise source.

Applying (4), the time derivative of the stochastic time shift is given by $\dot{\theta}(t) = -v_{12}^T(y)(1/C)i_w(t)$, where $v_{12}^T(y)$ stands for the second component of the vector $v_1^T(y)$. Thus, the sensitivity to noise is given by the time-varying coefficient $c_w(t) = -v_{12}^T(y)(1/C)$. In Fig. 2(a), $c_w(t)$ is compared with the TP-ISF, calculated from (5). The good agreement shows that, in this case, the amplitude-dependent term, neglected in the TP-ISF calculation, has a minor influence on the system response. The result from harmonic balance $(1/\omega_o)b_w(t)$ [obtained through (25)] has also been superimposed (with diamonds) and is overlapped with $c_w(t)$. The approximately zero average value of these two calculations and the ISF indicates that the parallel oscillator, with a parallel current noise source, has no phase noise due to the $1/f$ noise.

Now a noise voltage source $e_\gamma(t)$, connected in series with the nonlinearity, is going to be considered. Application of the

TABLE I
CUBIC NONLINEARITY OSCILLATOR WITH VOLTAGE NOISE SOURCE. COMPARISONS OF PHASE-NOISE SENSITIVITIES

	Floquet analysis: $\omega_0 c_w(t)$, $\omega_0 c_\gamma(t)$	ISF $\omega_0\Gamma(\omega_o t)$	Harmonic-balance: $b_w(t)$, $b_\gamma(t)$
Noise close to DC	DC term: 1.9796 10^9 rad/(v·s)	DC term: 1.4614 10^9 rad/(v·s)	DC term: 1.9804 10^9 rad/(v·s)
Noise close to first harmonic	First harmonic: 3.6396 10^9 rad/(v·s)	First harmonic: 2.8859 10^9 rad/(v·s)	First harmonic: 3.6466 10^9 rad/(v·s)
Noise close to second harmon	Second harmonic: 1.5927 10^9 rad/(v·s)	Second harmonic: 1.3815 10^9 rad/(v·s)	Second harmonic: 1.5694 10^9 rad/(v·s)
Noise close to third harmonic	Third harmonic: 2.2838 10^9 rad/(v·s)	Third harmonic: 1.6673 10^9 rad/(v·s)	Third harmonic: 2.2734 10^9 rad/(v·s)

tangential ISF formulation, in its present form [5], would be impossible. This source location requires the derivative of the nonlinearity with respect to the noise source (instead of the factor $1/C$ or $1/L$). Thus, the factor to be used in (8) is $g(v_o(t)) = (1/C)(a + 2mv_o(t) + 3bv_o(t)^2)$, which depends on the oscillator steady state. Fig. 2(b) shows the comparison between the results provided by (4) and (8). There is still a very good agreement. The calculation from harmonic balance $(1/\omega_o)b_\gamma(t)$ (26) has been superimposed and is again overlapped with $c_\gamma(t)$.

In Table I, numerical comparisons are presented of the sensitivity to noise provided by each of the two time-domain techniques and the one provided by the frequency-domain expression (15).

VI. CONCLUSIONS

An analytical comparison between different phase-noise analysis methods has been carried here with the aim to clarify the relationships existing between them. A frequency-domain approach to the perturbed differential equation of the free-running oscillation has enabled showing the excellent agreement between the phase-noise calculation, based on Floquet's analysis, and the carrier-modulation approach. The approximate analytic calculation of the ISF, based on TP-ISF, has good qualitative agreement with the two former techniques. The principles and formulation of the TP-ISF technique are close to those of the Floquet's analysis and this formulation has the advantage of not requiring the Jacobian matrix of the nonlinear-differential equation, which simplifies its application. However, in the phase and amplitude definition that is used, the amplitude fluctuations affect the phase fluctuations, and this is not taken into account, which leads to slightly less accurate results. Comparisons with the numerical and more accurate calculation of the ISF have not been carried out. The possibility to apply, through standard harmonic balance, the stochastic characterization of phase noise, based on Floquet's analysis, has also been shown. In addition to these analytical demonstrations, some numerical calculations, in a cubic-nonlinearity oscillator, have been presented.

APPENDIX

For a free-running oscillator, the HB equation can be written (in a very general way)

$$\overline{H} = [A(\omega_o)]\overline{X} + [B(\omega_o)]\overline{N}\left(\overline{X}\right) + [G_b]\overline{E}_b = \overline{0} \quad \text{(A.1)}$$

where $\overline{N}$ is the vector of nonlinear elements, $\overline{E}_b$ is the vector of bias generators, and $[A]$, $[B]$ and $[G_b]$ are linear matrices, with respective orders $l_x \times l_x$, $l_x \times l_n$ and $l_x \times l_b$. These matrix orders are generally different from those in (15). In (A.1), fewer state variables are used [11] since the time derivatives, which, in (15), form part of the state variable set, become powers of $j\omega$ in the standard equation (A.1). Considering noise generators $\overline{E}(t)$, the perturbed harmonic-balance system can be written (in a very general way) [13]

$$\begin{aligned}&\left\{[A(\omega_o)] + [B(\omega_o)]\left[\frac{\partial \overline{N}}{\partial \overline{X}}\right]_o\right\}\Delta\overline{X}(y)\\ &+\left\{\left[\frac{\partial A}{\partial \omega}\right]_o \overline{X}_o + \left[\frac{\partial B}{\partial \omega}\right]_o \overline{N}_o\right\}\Delta\omega_o(t)\\ &+\left\{\left[A\left\{\left(1+\frac{\Delta\omega_o}{\omega_o}\right)D_y\right\}\right]\right.\\ &\left.+\left[B\left\{\left(1+\frac{\Delta\omega_o}{\omega_o}\right)D_y\right\}\right]\left[\frac{\partial \overline{N}}{\partial \overline{X}}\right]_o\right\}\Delta\overline{X}(y)\\ &=[G]\left[e^{-jk\omega_o\theta}\right]\overline{E}(t) \qquad \text{(A.2)}\end{aligned}$$

where D_y is the derivation operator. Only first-order terms in the noise-source contributions have been considered. As in (12), the matrix containing the exponential terms $e^{-jk\omega_o\theta}$ is due to the phase shift between the temporal variable y used in the Fourier series expansions of $\overline{X}$ and $\overline{N}(\overline{X})$ and the time t in the noise source. Note that the inclusion of the term $[\partial G/\partial\omega_o]\Delta\omega_o$ would mean considering second-order effects in the perturbation equation [due to the smallness of $\overline{E}(t)$], thus, this term is neglected. Assembling terms in (A.2), it is possible to write

$$\begin{aligned}\left[\frac{\partial \overline{H}}{\partial \omega_o}\right] &= \left[\frac{\partial A}{\partial \omega_o}\right]\overline{X}_o + \left[\frac{\partial B}{\partial \omega_o}\right]\overline{N}_o\\ [JH]_o &= [A(\omega_o)] + [B(\omega_o)]\left[\frac{\partial \overline{N}}{\partial \overline{X}}\right]_o\\ \left[D\left(\overline{X}_o,\, \omega_o\right)\right] &= \left[A\left\{\left(1+\frac{\Delta\omega_o}{\omega_o}\right)D_y\right\}\right]\\ &+\left[B\left\{\left(1+\frac{\Delta\omega_o}{\omega_o}\right)D_y\right\}\right]\left[\frac{\partial \overline{N}}{\partial \overline{X}}\right]_o \qquad \text{(A.3)}\end{aligned}$$

where $[D(\overline{X}_o,\, \omega_o)]$ is a matrix operator performing derivatives with respect to the y variable. Thus, the perturbed harmonic-balance equation becomes

$$\begin{aligned}&[JH]_o\Delta\overline{X} + \left[\frac{\partial \overline{H}}{\partial \omega_o}\right]\Delta\omega_o + \left[D\left(\overline{X}_o,\, \omega_o\right)\right]\Delta\overline{X}\\ &\qquad = [G]\left[e^{-jk\omega_o\theta}\right]\overline{E}(t). \qquad \text{(A.4)}\end{aligned}$$

Now the kernel $\overline{V}_1$ of $[JH]_o^+$ is calculated. Imposing the normalization condition $\overline{V}_1^+[\partial\overline{H}/\partial\omega_o] = 1$, and solving for $\Delta\overline{X}$ fulfilling $\overline{V}_1^+[D(\overline{X}_o,\, \omega_o)]\Delta\overline{X} = \overline{0}$, the left-hand-side multiplication of (A.4) by $\overline{V}_1^+$ provides the same expression (19). Note that (A.4) must be fulfilled for all the possible $\Delta\overline{X}$ (which has an undetermined component), and, in particular, for the one fulfilling $\overline{V}_1^+[D(\overline{X}_o,\, \omega_o)]\Delta\overline{X} = \overline{0}$. This simplifies the calculation and confirms the validity of (19) for a general harmonic-balance formulation. At the time of determining the amplitude perturbation (and from it, the amplitude noise), a suitable approximation of the operator $[D(\overline{X}_o,\, \omega_o)]$ should be used. Due to the slow variations of $\Delta\overline{X}$, compared to the other terms of (A.4), the influence of $[D(\overline{X}_o,\, \omega_o)]\Delta\overline{X}$ will be negligible in most cases.

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